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Fuzzy ARX Modeling of Dynamic Systems

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Abstract— **Studies on the effectiveness of fuzzy logic for nonlinear modeling are presented. Although successful applications of fuzzy logic in many diverse areas are reported in the literature, commonly the explicit description of the key features of fuzzy logic yielding these outcomes is not addressed. The present research addresses this issue, to understand the basic features of fuzzy logic in nonlinear modeling applications in the context of dynamic system modeling, data modeling, signal modeling etc. This is accomplished by stochastic inputs to a nonlinear system modeled by fuzzy logic. The modeling performance is investigated in relation to the degree of nonlinearity of the model and the probability density of the stochastic inputs.**

I. INTRODUCTION

Although successful applications of fuzzy logic in many diverse areas are reported in the literature, the explicit description of the key features of fuzzy logic yielding these outcomes is generally not addressed. Naturally, it is understandable in such a case there is no much motivation to investigate further if the approach is satisfactory enough for the goal. This situation is easy to realize if one considers that there is still no systematic method of assignment of precise number of fuzzy membership functions, their shapes and locations with respect to the application at hand. Therefore, data driven fuzzy logic modeling is one of the essential approaches to determine such assignments in a pragmatic way. This is fully justified if one also considers that pragmatic solutions to soft issues is the main mission of soft computing.. Having pointed out this peculiarity, the present research aims for investigate the fuzzy logic performance in modeling applications to gain insight into the capabilities of fuzzy logic in dealing with nonlinear cases. Being one of the main components of soft computing, fuzzy logic deals with nonlinearity characteristics of the issues of soft computing allowing for uncertainties, vagueness which are expectedly inherent to the case in question. One important category of fuzzy logic methods is the Takagi-Sugeno [1,2] (TS) type fuzzy modeling where piecewise local linear models are used to represent the output domain which is referred to as the universe of discourse of consequents in the fuzzy logic terminology. In this work also TS type fuzzy logic is considered as the approach do not require defuzzification process as such a process is required in the Mamdani type [3,4] fuzzy logic counterpart. The organisation of the paper is as follows. Section two briefly describes the data driven TS fuzzy modeling. Section three deals with both a dynamic system model subject to fuzzy logic modeling in the way as described in section two and its statistical properties with respect to stochastic inputs. The stochastic inputs provide the data to establish the dynamic model. Section four presents the computer experiments. This is followed by conclusions in section five.

II. FUZZY MODELING

A. Takagi-Sugeno Fuzzy Modeling

Takagi-Sugeno (TS) type fuzzy modeling consists of set of fuzzy rules a local input-output relation in a linear form as

$$\begin{aligned} &R_i : If \quad x_1 \ \ is \ \ A_{i1} \ \ and \ldots x_n \ \ is \ A_{in} \\ &\quad Then \ \ \hat{y}_i = a_i x + b_i \,, \qquad i = 1,2,\ldots\ldots,K \end{aligned} \tag{1}$$

where R_i is the ith rule, $\boldsymbol{x}=[x_1,x_2,\ldots\ldots,x_n]^T \in X$ is the vector of input variables; $A_{i1}, A_{i2},\ldots,A_{in}$ are fuzzy sets and y_i is the rule output; K is the number of rules. The output of the model is calculated through the weighted average of the rule consequents of the form

$$\hat{y} = \frac{\sum_{i=1}^{K} \beta_i(x)\,\hat{y}_i}{\sum_{i=1}^{K} \beta_i(x)} \tag{2}$$

In (2), $\beta_i(x)$ is the degree of activation of the ith rule

$$\beta_i(x) = \prod_{j=1}^{n} \mu_{A_{ij}}(x_j), \quad i = 1,2,\ldots.,K \tag{3}$$

where $\mu_{A_{ij}}(x_j)$ is the membership function of the fuzzy set A_{ij} at the input (antecedent) of R_i. To form the fuzzy system model from the data set with N data samples, given by

$$X=[\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots\ldots\ldots,\boldsymbol{x}_N]^T, \;\; Y=[y_1, y_2, \ldots\ldots, y_N]^T \tag{4}$$

where each data sample has a dimension of n (N>>n). First the structure is determined and afterwards the parameters of the structure are identified. The number of rules characterizes the structure of a fuzzy system. The number of rules is determined by clustering methods. Fuzzy clustering in the Cartesian product-space $X \times Y$ is applied to partition the training data. The partitions correspond to the characteristic regions where the system's behaviour is approximated by local linear models in the multidimensional space.

Given the training data Γ and the number of clusters K, a suitable clustering algorithm [5] is applied. One of such clustering algorithms is known as Gustafson-Kessel (GK)

[6]. As result of the clustering process a fuzzy partition matrix U is obtained. The fuzzy sets in the antecedent of the rules are identified by means of the partition matrix U which has dimensions $[N\times K]$; N is the size of the data set. The ik-th element of $\mu_{ik}\in[0,1]$ is the membership degree of the i-th data item in cluster k; that is, the i-th row of U contains the point wise description of a multidimensional fuzzy set. One-dimensional fuzzy sets A_{ij} are obtained from the multidimensional fuzzy sets by projections onto the space of the input variables x_j. This is expressed by the point-wise projection operator of the form [7,8]

$$\mu_{A_{ij}}(x_{jk}) = proj_j(\mu_{ik}) \tag{5}$$

The point-wise defined fuzzy sets A_{ij} are then approximated by appropriate parametric functions. The consequent parameters for each rule are obtained by means of linear least square estimation. For this, consider the matrices $X=[\mathbf{x}_1,\ldots,\mathbf{x}_N]^T$, $X_e[X,\mathbf{1}]$ (extended matrix $[N\times(n+1)]$) ; Λ_i (diagonal matrix dimension of $[N\times N]$) and

$$X_E=[(\Lambda_1 X_e);\ (\Lambda_2 X_e);\ldots\ (\Lambda_K X_e)]\quad ([N\times K(n+1)]) \tag{6}$$

where the diagonal matrix Λ_i consists of normalized membership degree as its k-th diagonal element

$$_n\beta_i(x_k) = \frac{\beta_i(x_k)}{\sum_{j=1}^{K}\beta_j(x_k)} \tag{7}$$

The parameter vector ϑ dimension of $[K\times(n+1)]$ is given by

$$\vartheta=[\vartheta_1^T\ \ \vartheta_2^T\ \ \ldots\ldots\ \ \vartheta_K^T]^T \tag{8}$$

where $\vartheta_i^T=[\mathbf{a}_i^T\ \ b_i]$ $(1\le i\le K)$.

Now, if we denote the input and output data sets as X_E and Y respectively then, the fuzzy system can be represented as a regression model of the matrix form

$$Y=X_E\,\vartheta+e \tag{9}$$

For a model with single output (9) becomes

$$y=X\,\vartheta+e \tag{10}$$

where

$$\vartheta^T=[\mathbf{a}^T\ \ b] \tag{11}$$

B. GK Clustering Algorithm

For clustering, there are several effective fuzzy clustering algorithms available. Gustafson-Kessel algorithm is the one, which is commonly used due to some desirable features. In order to detail the modeling enhancement achieved in this work, GK algorithm is briefly explained below.

GK Algorithm: Given the data Z with the number of data samples N, number of clusters M, fuzziness parameter m>1, and the termination tolerance $\varepsilon>0$, initialize the fuzzy partition matrix $U=[\mathbf{u}_i](i=1,2,..M)$, randomly. Then

Repeat for $l=1,2,\ldots$

Step1: Compute cluster prototypes

$$v_i^{(l)} = \frac{\sum_{k=1}^{N}(u_{ki}^{(l-1)})^m z_k}{\sum_{k=1}^{N}(u_{ki}^{(l-1)})^m},\quad 1\le i\le M \tag{12}$$

Step 2: Compute covariance matrices:

$$F_i = \frac{\sum_{k=1}^{N}(u_{ki}^{(l-1)})^m (z_k - v_i^{(l)})(z_k - v_i^{(l)})^T}{\sum_{k=1}^{N}(u_{ki}^{(l-1)})^m}, 1\le i\le M \tag{13}$$

Step 3: Compute distances to cluster prototypes:

$$d_{ki}^2 = (z_k - v_i^{(l)})^T D_i (z_k - v_i^{(l)}), 1\le i\le M, 1\le k\le N \tag{14}$$

where

$$D_i = [\det(F_i)^{1/(n+1)} F_i^{-1}] \tag{15}$$

which is called *norm-inducing matrix*.

Step 4: Update the partition matrix:

$$\begin{aligned} &for\ \ 1\le i\le M, 1\le k\le N \\ &if\ \ d_{ki}>0 \\ &u_{ki}^{(l)} = \frac{1}{\sum_{j=1}^{M}(d_{ki}/d_{kj})^{2/(m-1)}} \\ &else\ \ if\ \ d_{ki}=0 \\ &u_{ki}^{(l)}=1 \end{aligned} \tag{16}$$

$$u_{ik}^{(l)}\in[0,1]\ \ with\ \ \sum_{i=1}^{M}u_{ik}^{(l)}=1 \tag{17}$$

until $\|U^{(l)}-U^{(l-1)}\|<\varepsilon$

III. Nonlinear Model and Statistical Properties

A. Dynamic System Model

For the investigation of fuzzy modeling with stochastic excitations, a nonlinear system

$$y(t) = 1 - e^{-x(t)/\tau} \tag{18}$$

is considered. Here $x(t)$ is the system variable. For a data driven fuzzy modeling approach, the system representation is cast into a recursive form as

$$y(t) = a(t)\,y(t-1) + u(t) \tag{19}$$

where the time varying AR coefficient $a(t)$ and the input $u(t)$ are given by

$$a = e^{-[x_2(t) - x_1(t)]/\tau}$$

$$u(t) = 1 - e^{-[x_2(t) - x_1(t)]/\tau} \qquad (20)$$

B. Statistical Properties

The probability density function (*pdf*) of the stochastic outputs of the fuzzy model can be computed from the *pdf* of the inputs. By studying both *pdf*s, one can obtain some information about the nature of the nonlinearity of the dynamic system. The *pdf* computations can be carried out as follows.

Consider the nonlinear dynamic system given by $y=g(x)$. To find $f_y(y)$ for a given x we solve the equation

$$y = g(x) \qquad (21)$$

for x in terms of y. If $x_1, x_2, ..., x_n, ..$ are all its *real* roots,

$$x_1=g(y_1) \quad x_2= g(y_2) = ... x_n= g(y_n) = ...$$

Then

$$f_y(y) = \frac{f_x(x_1)}{|g'(x_1)|} + ... + \frac{f_x(x_2)}{|g'(x_2)|} + ... + \frac{f_x(x_n)}{|g'(x_n)|} + .. \qquad (22)$$

Clearly, the numbers $x_1, x_2, ..., x_n, ..$ depend on y. If, for a certain x, the equation $x= g(y)$ has no real roots, then $f_y(y) = 0$.

According the theorem above, consider the nonlinear dynamic system given by (18). Then,

$$y = g(x) = 1 - e^{-x/\tau} \qquad (23)$$

$$x_1 = \ln\left(\frac{1}{1-y}\right)^{\tau} \qquad (24)$$

$$g'(x) = \frac{1-y}{\tau} \qquad (25)$$

$$f_y(y) = \frac{\tau}{1-y} f_x(x_1) \qquad (26)$$

If we assume uniform density between 0 and 1, for $f_x(x)$, the pdf of the system output is

$$f_y(y) = \frac{\tau}{1-y} \qquad \left(0 \le y \le 1 - \exp(1/\tau)\right) \qquad (27)$$

which satisfies

$$\int_0^{1-e^{1/\tau}} f_y(y)\,dy = 1$$

The same computations for input with Gaussian pdf with a shift of x_o yields

$$f_y(y) = \frac{\tau}{\sqrt{2\pi}\,\sigma} \frac{1}{1-y}\; e^{-\frac{1}{2}\left[\frac{\ln\left(\frac{1}{1-y}\right)^{\tau} - x_o}{\sigma^2}\right]^2} \qquad (28)$$

The variation of $f_y(y)$ given in (27) and (28) are sketched in figures 8 and 9.

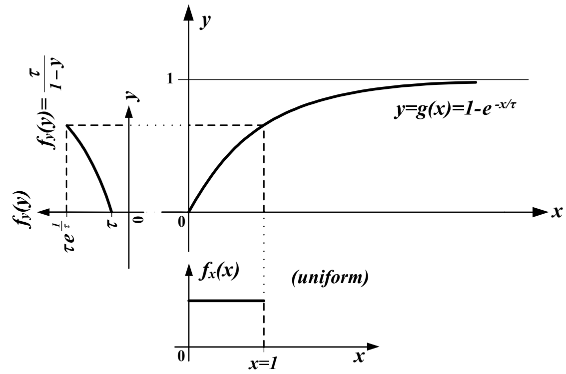

Fig. 1 Uniform probability density functions (pdf) at the model input and and the ensuing pdf at the model output

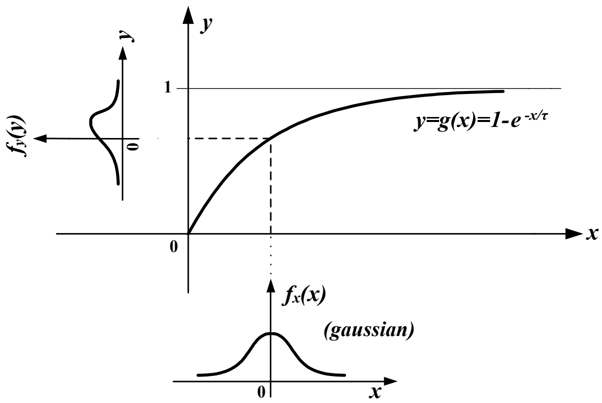

Fig.2. Gaussian probability density function (pdf) at the model input and the ensuing pdf at the model output

The pdf of $u(t)$ given by (20) is computed as follows.

$$u(t) = 1 - e^{-[x_2(t) - x_1(t)]/\tau} \qquad (29)$$

We define a new variable w as $w = x_2 - x_1$.

$$f_w(w) = \int_{-\infty}^{+\infty} f_{x_1}(w - x_2) f_{x_2}(-x_2)\,dx_2$$

We assume $x1$ and $x2$ have uniform density as this was the case in this research, $f_w(w)$ is obtained as seen in figure 9

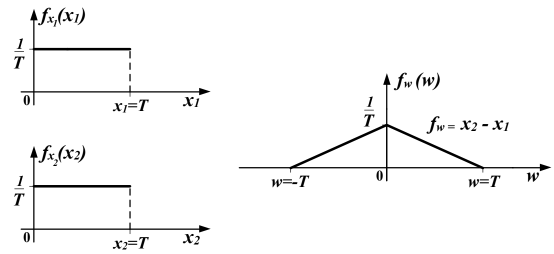

Fig. 3. Probability density function (pdf) of a random variable, which represents the difference of two other random variables with uniform pdfs.

where

$$f_w(w) = -\frac{w}{T^2} + \frac{1}{T} \quad (w > 0)$$

$$f_w(w) = \frac{w}{T^2} + \frac{1}{T} \quad (w < 0) \qquad (30)$$

Using the theorem (22) on the pdf of a function of a random variable, we write

$$u(w) = 1 - e^{-w/\tau} \tag{31}$$

$$u(w) = g(w) = 1 - e^{-w/\tau} \tag{32}$$

The root of (32) is

$$w_1 = \ln\left(\frac{1}{1-u}\right)^{\tau} \tag{33}$$

The derivation of g(w) is given by

$$g'(w) = \frac{1}{\tau} e^{-w/\tau} = \frac{1}{\tau}(1-u) \tag{34}$$

so that, using the theorem (22), for $u \le 0$

$$f_{-u}(u) = \frac{f_w(w_1)}{|g'(w_1)|} = \frac{\frac{\tau}{T^2}\ln\left(\frac{1}{1-u}\right)^{\tau} + \frac{\tau}{T}}{1-u} \tag{35}$$

and for $u \ge 0$

$$f_{+u}(u) = \frac{f_w(w_1)}{|g'(w_1)|} = \frac{-\frac{\tau}{T^2}\ln\left(\frac{1}{1-u}\right)^{\tau} + \frac{\tau}{T}}{1-u} \tag{36}$$

are obtained, so that

$$\int_{1-e^{T/\tau}}^{0} f_{-u}(u)\, du + \int_{0}^{1-e^{-T/\tau}} f_{+u}(u)\, du = 1 \tag{37}$$

The input u(t) to nonlinear system is seen in figures 2 and 3. The same calculations for the time varying autoregressive (AR) model coefficient a in (19) yields, for a ≤ 1, we obtain

$$f_{a1}(a) = \frac{\tau}{T\,a}\left[\frac{1}{T}\ln\left(\frac{1}{a}\right)^{\tau} + 1\right] \tag{38}$$

and for $a \ge 1$, we obtain

$$f_{a2}(a) = \frac{\tau}{T\,a}\left[-\frac{1}{T}\ln\left(\frac{1}{a}\right)^{\tau} + 1\right] \tag{39}$$

so that

$$\int_{e^{-T/\tau}}^{1} f_{a1}(a)\, da + \int_{1}^{e^{T/\tau}} f_{a2}(a)\, da = 1 \tag{40}$$

The pdfs of u and a are shown in figures 4 and 5, respectively.

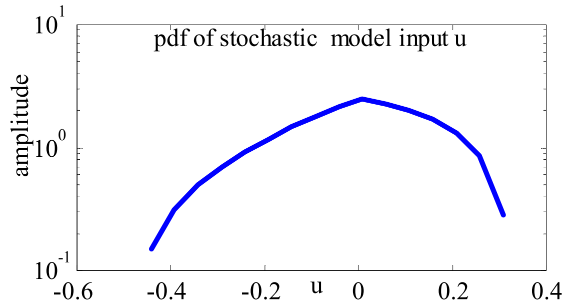

Fig. 4. Pdf of the model input u for T=10 and τ=25

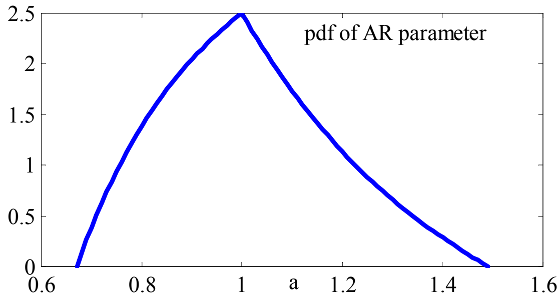

Fig. 5. Pdf of autoregressive model parameter ***a*** for $T=10$ and $\tau=25$

IV. Computer Experiments

Throughout the experiments the width of the uniform density (T) shown in figure 3, is kept constant as $T=10$. The experiments with different τ values are presented below.

A. Experiments with $\tau=25$ and 2 fuzzy sets

For fuzzy modeling, first the system variable $x(t)$ is considered as band limited white noise as shown in figure 3 and the system response is obtained from (19) for 150 samples. Based on this data the TS fuzzy model of the system is established for two clusters, i.e. two local models. The function $y=f(x)$ describing the nonlinear system is shown in figure 6 for $\tau=5$, $\tau=10$ and $\tau=25$. The data driven local models from the clusters are shown in figure 7. The system membership functions and the system performance are shown in figure 8 where in the lower plot the true model output and the fuzzy model output are shown together.

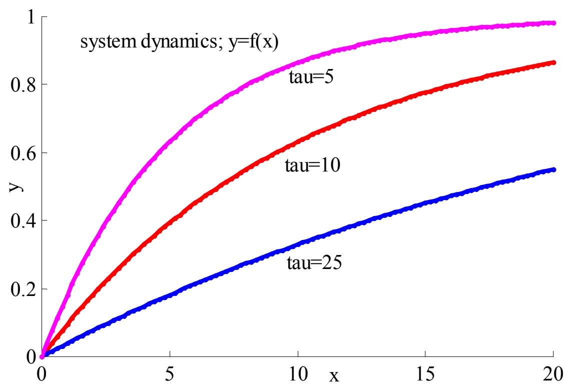

Fig. 6. The nonlinear dynamic system

There is some difference between these outputs and this is constructive for generalization capability of the model for unknown (test) inputs.

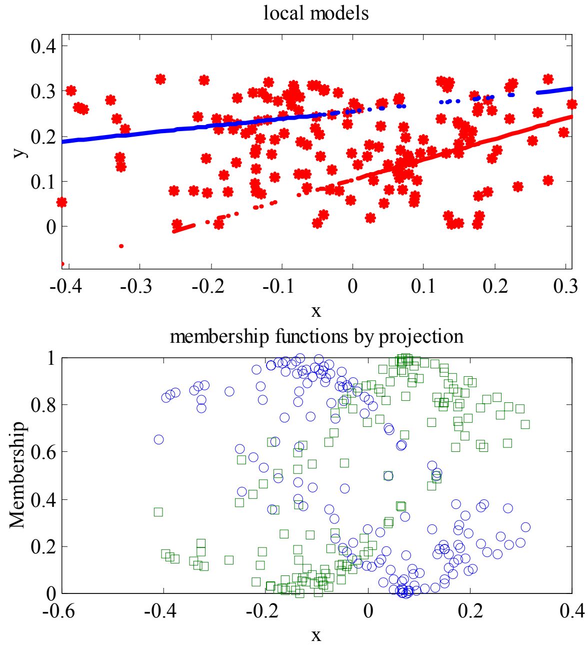

Fig. 7. Fuzzy clustering and ensuing data driven local models

Figure 9 represents the model performance for the test data where the true and the estimated model outputs are shown together in the uppermost plot. Correspondingly the model inputs corresponding to these outputs are shown in the figure as middle and lower plots, respectively. From figures 1 and 3, it is seen that the fuzzy model has satisfactory performance.

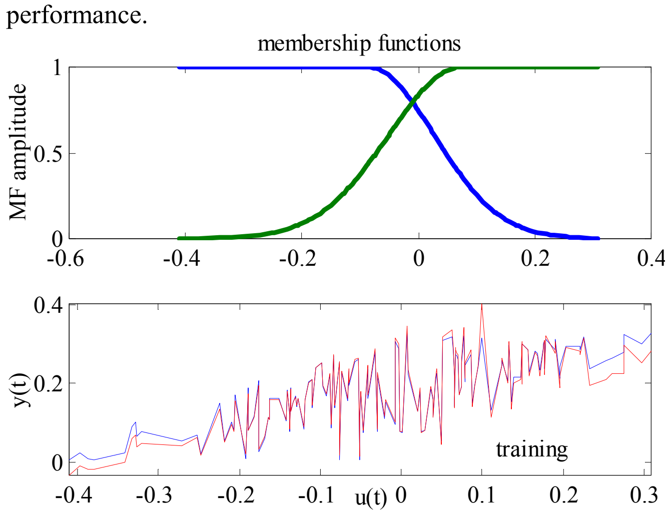

Fig. 8. Membership functions (upper) and the fuzzy model outputs (lower) as the true model outputs and the estimated counterparts.

The system variable *x(t)* used to form the model are from band-limited white noise. The system test inputs are from the perception measurements from a virtual agent reported elsewhere [9]. One notes that, the test *x(t)* inputs to the system has wide frequency range. However, the nonlinear system behaves a nonlinear low-pass filter so that three local models give satisfactory estimated system outputs matching the true counterparts rather satisfactorily. From the pdf characteristic shown in figure 4, it is clear that the relatively significant deviations at the upper and lower limits of *u(t)* are observed in figure 9.

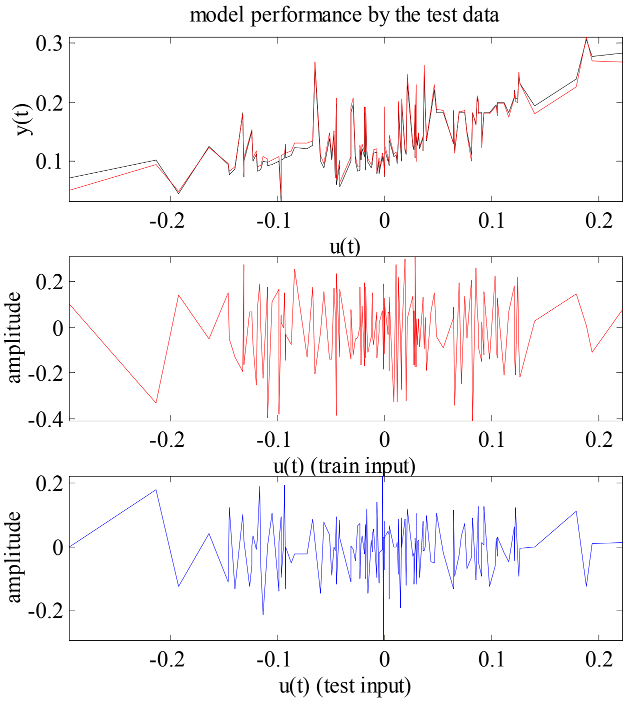

Fig. 9. True model output and its estimation by fuzzy modeling (uppermost plot); the input to nonlinear system used for fuzzy model formation (middle); the input to nonlinear system used for testing the fuzzy model performance (lowermost).

B. Experiments with τ=10 and 2 fuzzy sets

For this case the pdf of the random variable *u* is shown in figure 10.

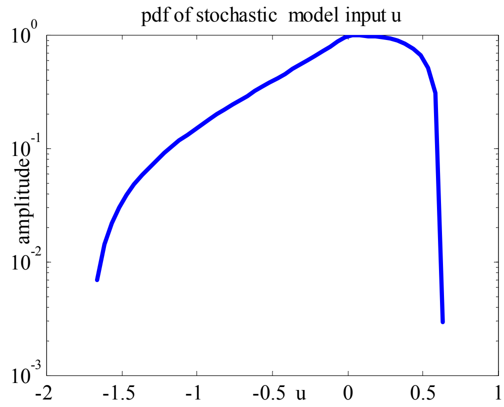

Fig. 10. Pdf of the model input u for T=10 and τ=10

The clustering and local linear models are shown in figure 11. Correspondingly, the membership functions and the model performance is shown in figure 12. Since the degree of nonlinearity is higher, the model performance for training is inferior to that obtained with τ*=25*. Consequently the same unfavourable situation occurs for the test case shown in figure 13. The higher differences between the true model output and the test output at the upper and lower ends of the variable *u* are explained by the pdf variation in figure 10.

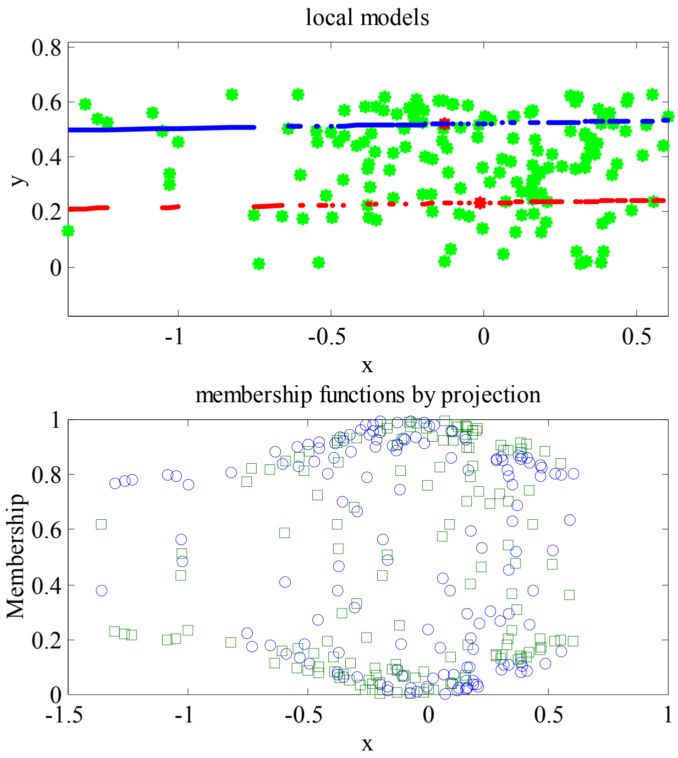

Fig. 11. Fuzzy clustering and ensuing data driven local models

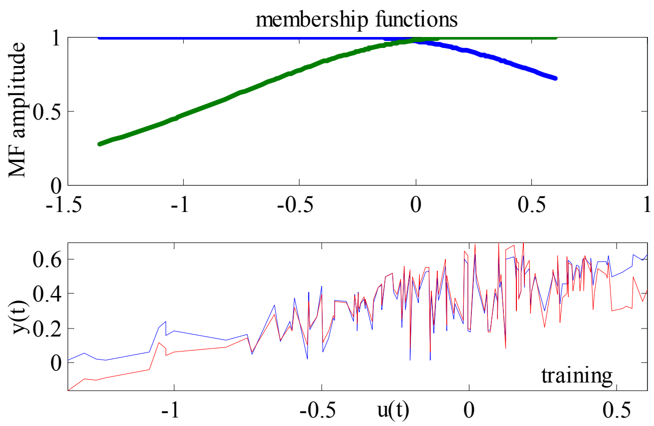

Fig. 12. Membership functions (upper) and the fuzzy model outputs (lower) as the true model outputs and the estimated counterparts.

C. Experiments with τ=10 and 4 fuzzy sets

To a certain extend, the modeling performance presented in figures 12 and 13 can be improved by increasing the number of membership functions since for τ=10, the system dynamics given in figure 10 can still be satisfactorily represented by two linear local models. To demonstrate this, the same experiments are repeated for four membership functions and the results are presented in figures 14-16. The slight improvement in model formation and the test cases are observed in figure 15 and 16 compared to the figures 12 and 13. The higher model errors at the lower and upper ends of the variable *u* are due to lower pdf values at these regions.

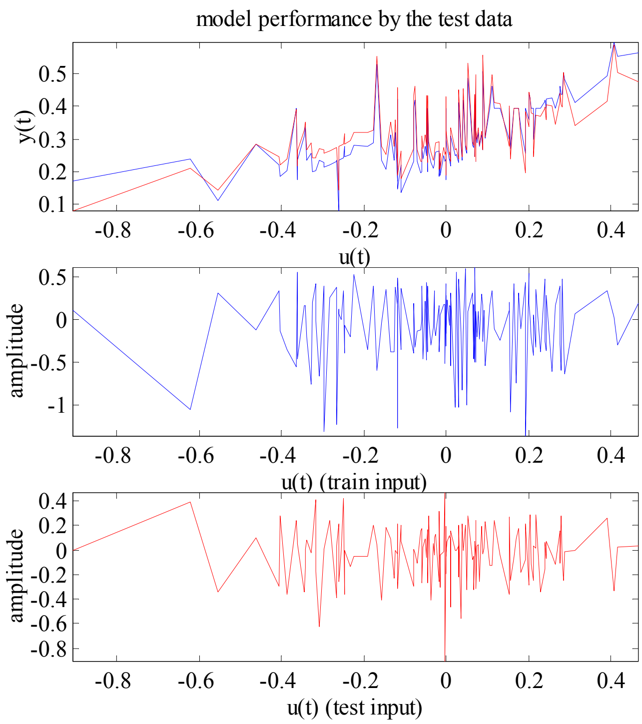

Fig. 13. True model output and its estimation by fuzzy modeling (upper); the input to nonlinear system used for fuzzy model formation (middle); the input to nonlinear system used for testing the fuzzy model performance (lower).

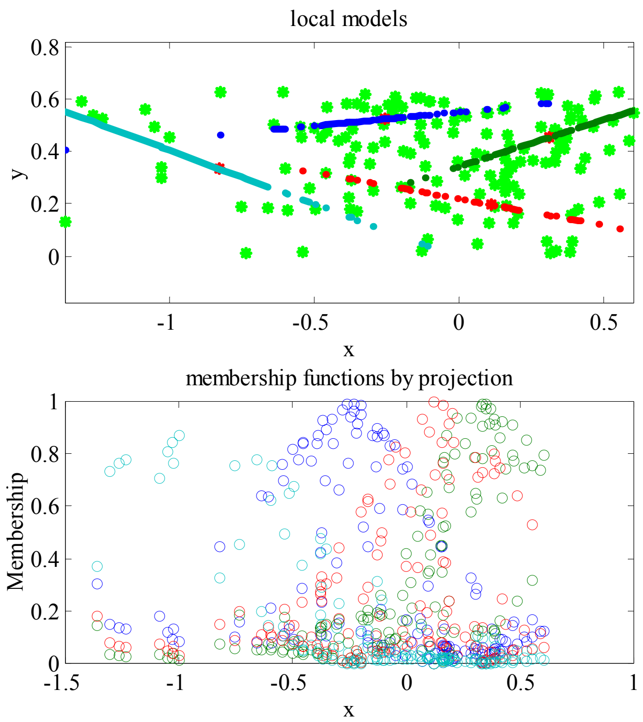

Fig. 14 . Fuzzy clustering and ensuing data driven local models

The improvement is due to better representation of the nonlinearity of the system dynamics with the increased number of fuzzy sets as there is no apparent redundancy of these sets as clearly seen in figure 15.

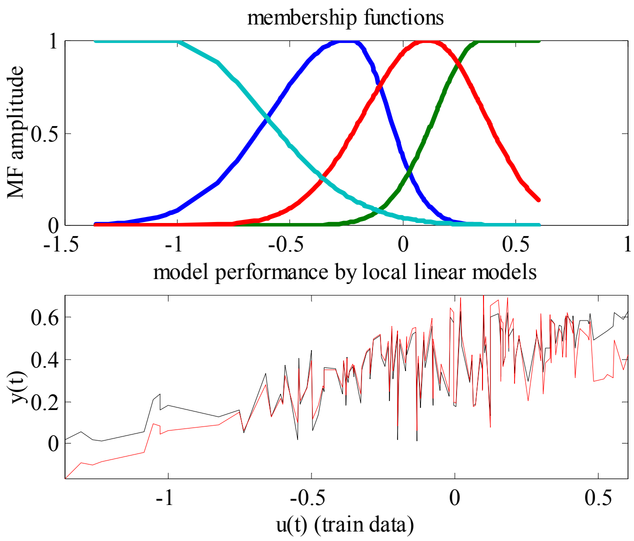

Fig. 15 Membership functions (upper) and the fuzzy model outputs (lower) as the true model outputs and the estimated counterparts.

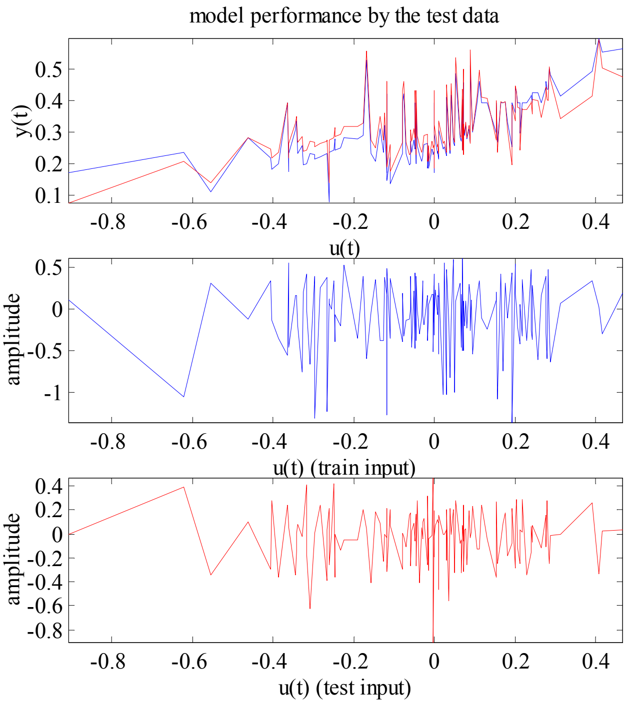

Fig. 16 Model performance by the test data: true model output and its estimation by fuzzy modeling (uppermost plot); the input to nonlinear system used for fuzzy model formation (middle); the input to nonlinear system used for testing the fuzzy model performance (lowermost).

D. Experiments with $\tau=5$ and 2 fuzzy sets

The experiments with the increased degree of nonlinearity taking $\tau=5$ are presented in figures 17, 18 and 20. Due to the increased degree of nonlinearity of the system dynamics, the deterioration of the system performance (as to figures 14-16) for the same experimental conditions is clearly verified. However, in these outcomes the shape of the pdf of the input u plays also important role; that is the concentration of the data samples around $u=0$ is easily observed. In accordance with this, the consistent establishment of the fuzzy sets, as result of clustering, is seen in figure 17.

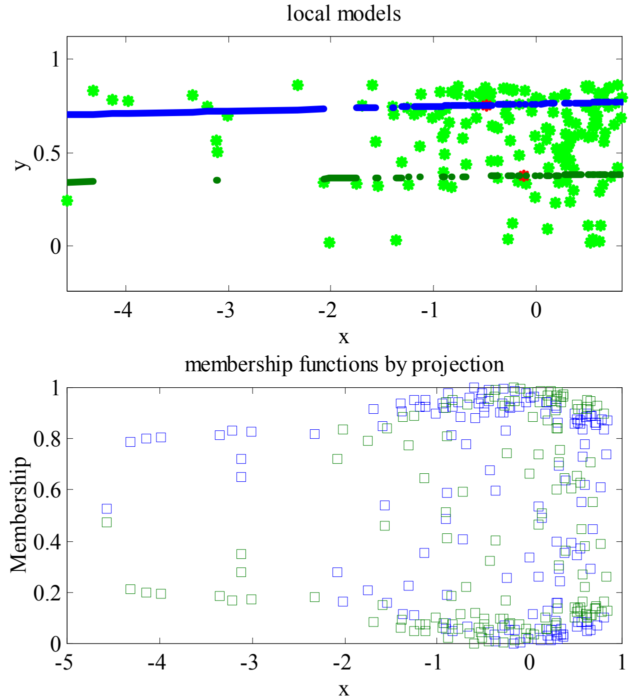

Fig. 17. Fuzzy clustering and ensuing data driven local models

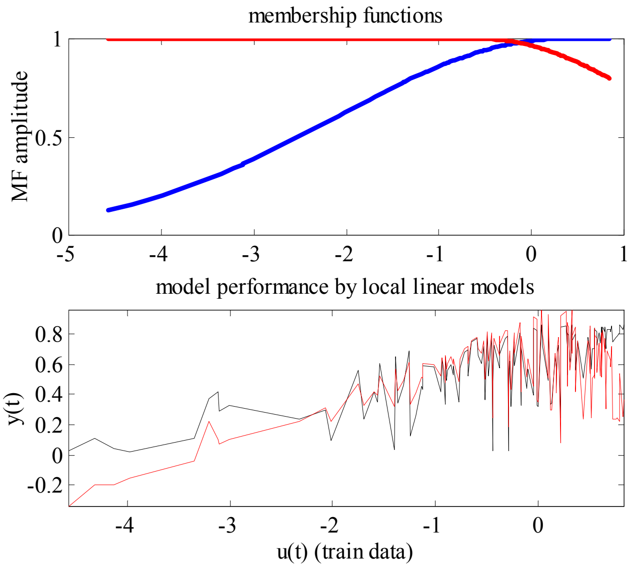

Fig. 18. Membership functions (upper) and the fuzzy model outputs (lower) as the true model outputs and the estimated counterparts.

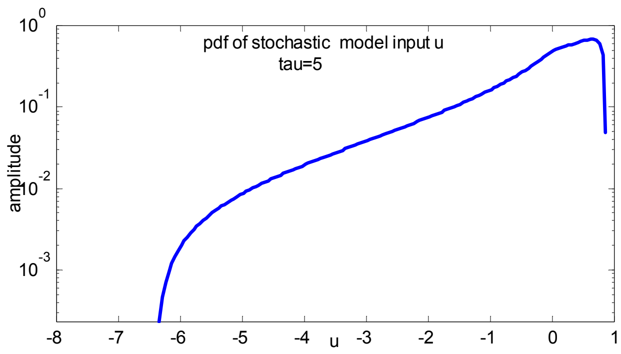

Fig. 19. Pdf of the model input u for $T=10$ and $\tau=5$

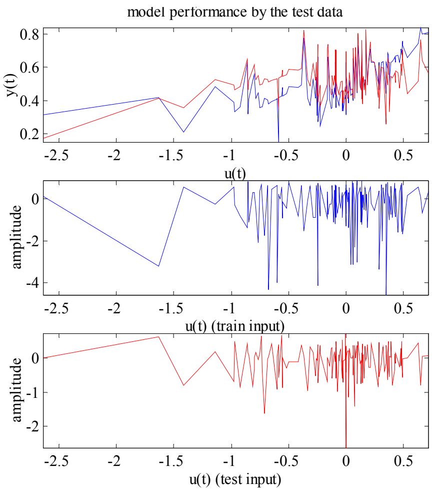

Fig. 20. Model performance by the test data: true model output and its estimation by fuzzy modeling (uppermost plot); the input to nonlinear system used for fuzzy model formation (middle); the input to nonlinear system used for testing the fuzzy model performance (lowermost).

V. Discussions and Conclusions

The analysis and identification of nonlinear systems from random data is one of the important topics of engineering systems to deal with complexity of modern man-machine interfaced systems. Among others, mention may be made to the design of fault tolerant systems, the application to fault diagnosis in building technological areas. Soft computing technology is a substantial supporting technology in such systems due to its acceptable solutions with pragmatic approximations. In this respect, TS fuzzy modeling is an essential means for representation of nonlinear dynamic systems for identification, control etc. Such system dynamics is represented relatively small number of fuzzy sets compared to other approaches, like an RBF network [10]. The important implication of this research is the utilisation of probabilistic methods for the analysis of nonlinear fuzzy models; namely, for fuzzy nonlinear dynamic system identification and modeling applications, where stochastic methods can play important cooperative role. The present research exemplifies how such methods can be implemented by using (band limited) white noise with uniform probability density and forming a fuzzy nonlinear model by the observed output. In particular, the dynamic system representation is a nonlinear AR model with eXogenous input so that it becomes ARX in the terminology of system identification. In this basic research, the system behaviour is fully explained by means of the probability densities of the input and observed outputs. It is purposely demonstrated how the fuzzy model performance is deteriorated by increasing degree of nonlinearity of the system and at what regions representations become comparatively poor. As an important component of soft computing paradigm, fuzzy logic is especially effective for dealing with nonlinearity. Therefore, the deterioration/poor representational regions can easily be eliminated by using appropriate stochastic inputs with desirable probability density according to the regions. This can be accomplished by addressing these regions for system response where the results are desired to be relatively more accurate. Since modeling is data driven approach, the number of data samples is important parameter.

Another important implication of this research can be seen in a different viewpoint. Referring to the growing interest in extending the theory of probability and statistics to allow for more flexible modeling of uncertainty, ignorance and fuzziness, the investigation of the properties of fuzzy modeling for statistical signals can be substantially lucrative in this respect. Comparatively, there is no much research reported in the literature on the area making explicit identification of statistical/stochastic fuzzy modeling properties where statistical/stochastic signals are in play in the context of fuzzy logic. In this respect, the present research is one such example with a basic endeavour involving fuzzy logic and stochastic signal cooperating in a modeling dynamic system modeling exercise.

Finally, effective associations of random data with fuzzy logic can be made and these associations can be exploited in variety of ways. This is especially the case in perception studies being carried out in the department at Delft, The Netherlands, where probability plays the essential role to model the perception process as described in reference 9. The work is based on an underlying probabilistic perception theory given in [11]. In these studies associations are made from visual perceptions of an environment to perceptual qualities of it. This is accomplished by simulating human vision interacting with the environment. The response is exponentially averaged. Because of the softness of perception, the association of probability with fuzzy logic is much demanded and the research progresses along that line, as well as others.

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