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Fuzzy Logic for Stochastic Modeling

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Exploring the growing interest in extending the theory of probability and statistics to allow for more flexible modeling of uncertainty, ignorance, and fuzziness, the properties of fuzzy modeling are investigated for statistical signals, which benefit from the properties of fuzzy modeling. There is relatively research in the area, making explicit identification of statistical/stochastic fuzzy modeling properties, where statistical/stochastic signals are in play. This research makes explicit comparative investigations and positions fuzzy modeling in the statistical signal processing domain, next to nonlinear dynamic system modeling.

1 INTRODUCTION

The concept *computing with words* is a fundamental contribution of fuzzy logic [1] to the paradigm of artificial intelligence (AI). Computing with words became feasible via the utilization of linguistic variables, where the words can be interpreted as semantic labels in relation to the fuzzy sets, which are the basic conceptual elements of fuzzy logic. Consequently, comprehensible computer representation of the domain issues can be created. On one side, dealing with fuzzy qualities quantitatively is a significant step in AI. On the other side, due to the same fuzzy qualities, the interpretability issues arise [2]. While fuzzy logic contributes to science in dealing with domain related fuzzy issues, it is natural to anticipate that fuzzy logic associated with the probability theory and statistics can better deal with fuzziness of the domain issues, spanning the exact sciences and the soft sciences.

The statistical aspects of fuzzy modeling have received relatively less attention than *computing with words* or *soft computing*. In dealing with the latter two aspects Mamdani type of fuzzy models are more convenient [3], addressing soft issues especially in soft domains. In contrast to this, the Takagi-Sugeno (TS) type fuzzy model [4] is presumably more convenient in engineering systems where the fuzzy logic consequents are local linear models rather than fuzzy sets. In this way, the defuzzification process is greatly simplified making fuzzy logic more pragmatic approach in applications where data-driven modeling is a natural choice. In this research stochastic signals with TS fuzzy modeling are considered, since such signals are rich in probabilistic and statistical information that can be exploited by means of fuzzy logic. In particular, the fuzzy model is considered as the representation of a general nonlinear dynamic system.

2 FUZZY MODELING

Takagi-Sugeno (TS) type fuzzy modeling [4] consists of a set of fuzzy rules as local input-output relations in a linear form thus:

$$R_i : If \quad x_1 \;\; is \;\; A_{i1} \;\; and \ldots x_n \;\; is \;\; A_{in} \qquad (1)$$
$$Then \;\; \hat{y}_i = a_i x + b_i \,, \qquad i = 1,2,\ldots\ldots, K$$

where R_i is the i-th rule, $x=[x_1,x_2,\ldots\ldots,x_n]^T \in X$ is the vector of input variables; A_{i1}, $A_{i2},\ldots,A_{in}$ are fuzzy sets and y_i is the rule output; K is the number of rules. The output of the model is calculated through the weighted average of the rule consequents, which gives

$$\hat{y} = \frac{\sum_{i=1}^{K} \beta_i(x)\,\hat{y}_i}{\sum_{i=1}^{K} \beta_i(x)} \qquad (2)$$

In (2), $\beta_i(x)$ is the degree of activation of the i-th rule

$$\beta_i(x) = \prod_{j=1}^{n} \mu_{A_{ij}}(x_j), \quad i = 1,2,\ldots., K \qquad (3)$$

where $\mu_{Aij}(x_j)$ is the membership function of the fuzzy set A_{ij} at the input (antecedent) of R_i. To form the fuzzy system model from the data set with N data samples, given by $X=[\mathbf{x}_1, \mathbf{x2}, \ldots\ldots\ldots,\mathbf{x}_N]^T$, $Y=[y_1, y_2, \ldots\ldots, y_N]^T$ where each data sample has a dimension of n ($N>>n$). First the structure is determined and afterwards the parameters of the structure are identified. The number of rules characterizes the structure of a fuzzy system. The number of rules is determined by clustering methods. Fuzzy clustering in the Cartesian product-space $X \times Y$ is applied for partitioning the training data. The partitions correspond to the characteristic regions where the system's behaviour is approximated by local linear models in the multidimensional space. Given the training data T and the number of clusters K, a suitable clustering algorithm [5] is applied. One of such clustering algorithms is known as Gustafson-Kessel (GK) [6]. As result of the clustering process a fuzzy partition matrix U is obtained. The fuzzy sets in the antecedent of the rules is identified by means of the partition matrix U which has dimensions $[N \times K]$, where N is the size of the data set and K is the number of rules. The ik-th element of $\mu_{ik} \in [0,1]$ is the membership degree of the i-th data item in cluster k; that is, the i-

th row of U contains the point wise description of a multidimensional fuzzy set. One-dimensional fuzzy sets A_{ij} are obtained from the multidimensional fuzzy sets by projections onto the space of the input variables x_j. This is expressed by the point-wise projection operator of the form $\mu_{Aij}(x_{jk})=proj_j(\mu_{ik})$ [7]. The point-wise defined fuzzy sets A_{ij} are then approximated by appropriate parametric functions. The consequent parameters for each rule are obtained by means of linear least square estimation. For this, consider the matrices $X=[x1,\ldots,xN]T$, $X_e[X,1]$ (extended matrix $[N\times(n+1)]$) ; Λi (diagonal matrix dimension of $[N\times N]$) and $X_E=[(\Lambda_1 Xe);(\Lambda 2X_e);\ldots\ldots(\Lambda KX_e)]$ $([N\times K(n+1)])$, where the diagonal matrix Λ_i consists of normalized membership degree as its k-th diagonal element

$$_n\beta_i(x_k)=\frac{\beta_i(x_k)}{\sum_{j=1}^{K}\beta_j(x_k)} \tag{4}$$

The parameter vector ϑ dimension of $[K\times(n+1)]$ is given by $\vartheta=[\vartheta_1 T\ \ \vartheta_2 T\ \ \ldots\ldots\ldots\ \vartheta_k T]T$ where $\vartheta iT=[aiT\ \ bi]$ $(1\leq i\leq K)$. Now, if we denote the input and output data sets as X_E and Y respectively, then the fuzzy system can be represented as a regression model of the matrix form $Y=X_E\ \vartheta+e$.

3 DYNAMIC SYSTEM MODELING

For the investigation of fuzzy modeling with stochastic excitations, a nonlinear system

$$y(t)=1-e^{-x(t)/\tau} \tag{5}$$

is considered. Here $x(t)$ is the system variable. For a data driven fuzzy modeling approach, the system representation is cast into a recursive form as

$$y(t)=a(t)\,y(t-1)+u(t) \tag{6}$$

where the time varying AR coefficient $a(t)$ and the input $u(t)$ are given by

$$a=e^{-[x_2(t)-x_1(t)]/\tau} \quad \text{and} \quad u(t)=1-e^{-[x_2(t)-x_1(t)]/\tau} \tag{7}$$

For fuzzy modeling, first the system variable $x(t)$ is considered as band limited white noise and the system response is obtained from (5) for 200 samples. Based on this data the TS fuzzy model of the system is established for three clusters, i.e. three local models. The membership functions and the system performance are shown in figure 1. In the lower plot the true model output and the fuzzy model output is shown together. There is some difference between these outputs and this is constructive for the generalization capability of the model for unknown (test) inputs. In the model τ is taken as $\tau=20$. Figure 2 represents the model performance for the test data. The true and the estimated model outputs are shown together in the upper plot.

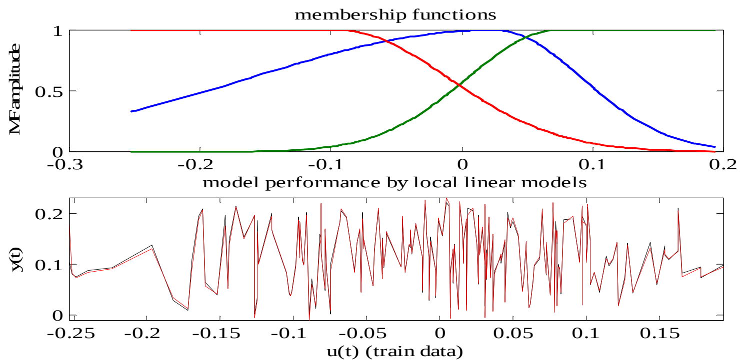

Fig. 1. Membership functions (upper) and the fuzzy model outputs (lower) as the true model outputs and the estimated counterparts

The model inputs corresponding to these outputs are shown in the figure as middle and lower plots, respectively. From figures 1 and 2 it is seen that the fuzzy model has satisfactory performance for stochastic inputs. The data samples of system variable *x(t),* which form the data-driven model, are from band-limited white noise. The system test input stems from perception measurements of a virtual agent reported elsewhere [8] where the present nonlinear system is representative of openness perception subject to measurement. It should be noted that, the test inputs to the system have wide frequency range. However, the nonlinear system behaves as a nonlinear low-pass filter so that three local models give satisfactory estimated system outputs, matching the true counterparts rather satisfactorily.

In order to investigate the pattern representation capabilities of fuzzy modeling a block of a time-series signal and its wavelet transform is considered. The time-series signal is a in particular band-limited white noise, and the number of clusters considered is five. For this case, the membership functions and the fuzzy model representation of the wavelet coefficients are shown in figure 3. Membership functions (upper) and the model outputs as true outputs and their estimated counterparts are also shown in the figure. The difference is significant due to the low number of fuzzy sets used for approximation. The above reported computer experiments show that TS fuzzy modeling is effective in modeling nonlinear dynamic systems and representation of patterns. In the nonlinear dynamic system representation, since the system is restricted to the lower frequency region, the Gaussian-shaped membership functions are capable of representing the system adequately. However in the pattern representation, since the frequency band is wide as the time-series data is band-limited white, in place of shaped Gaussian membership functions, the membership functions obtained directly from the clustering process are used. Otherwise, the shaped Gaussians are not enough narrow to represent the local variations. In other words, the local variations can not be represented by a restricted number of local linear models defined by the number of clusters.

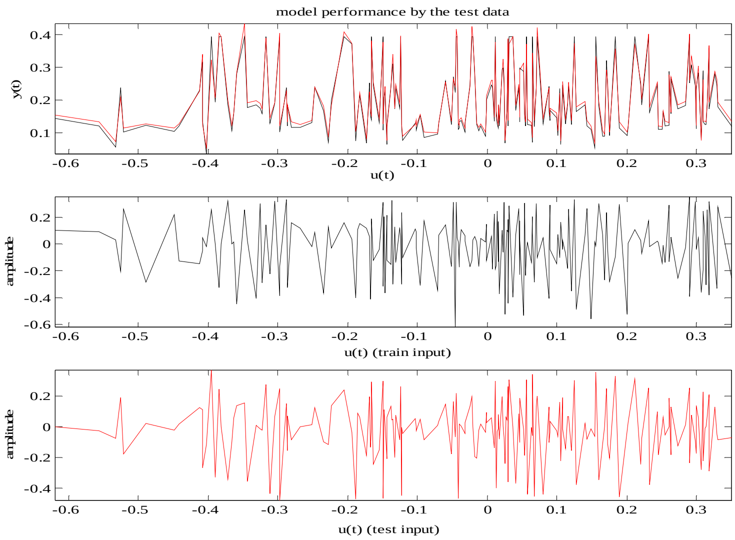

Fig. 2. True model output and its estimation by fuzzy modeling (upper); input to nonlinear system used for fuzzy model formation (middle); input to nonlinear system used for testing fuzzy model performance (lower)

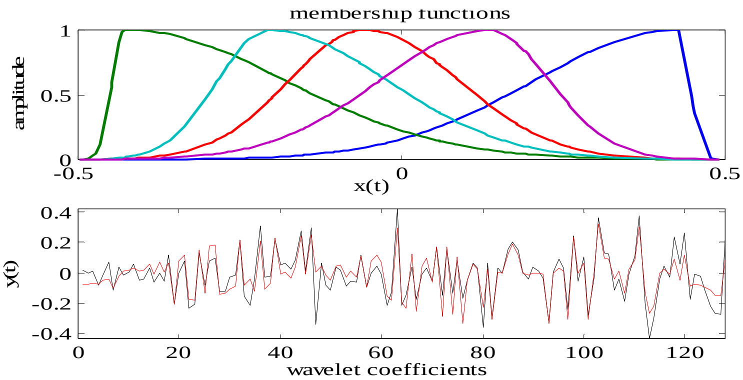

Fig. 3. Membership functions (upper) and model outputs as true outputs and their estimated counterparts involving five fuzzy sets

A similar situation obtains is the case of multivariable fuzzy modeling, where the membership functions are directly employed from the clustering. In this case the cause is different but the consequence is the same. Namely, there is an irrecoverable projection error due to projected and shaped membership functions from the clustered data, which prevents accurate representation of the dynamic model in multidimensional space [9]. By considering such basic features of fuzzy modeling, the fuzzy logic can be conveniently associated with the probabilistic entities, as this is stochastic signals and patterns, in this work.

4 PROBABILITY DENSITY FUNCTIONS

The probability density function (pdf) of the stochastic outputs of the fuzzy model can be computed from the pdf of the inputs. By studying both pdfs, one can obtain important information about the nature of the nonlinearity of the dynamic system. The pdf computations can be carried out as follows. Consider the nonlinear dynamic system given by *y=g(x)*. To find $f_y(y)$ for a given *x* we solve the equation *y= g(x)* for *x* in terms of *y*. If x_1 , x_2 ,…., x_n , .. are all its *real* roots, *x1=g(y1)* *x2= g(y2) =…… xn= g(yn) = ….* Then

$$f_y(y) = \frac{f_x(x_1)}{|g'(x_1)|} + ... + \frac{f_x(x_2)}{|g'(x_2)|} + ... + \frac{f_x(x_n)}{|g'(x_n)|} + .. \tag{8}$$

According to the theorem above, we consider the nonlinear dynamic system given as (5).

$$y = g(x) = 1 - e^{-x/\tau} \quad \text{and} \quad f_y(y) = \frac{\tau}{1-y} f_x(x_1) \tag{9}$$

Then, if we assume uniform density between 0 and 1, for $f_x(x)$, the pdf of the system output is

$$f_y(y) = \frac{\tau}{1-y} \qquad \left(0 \le y \le 1 - \exp(1/\tau)\right) \tag{10}$$

which satisfies $\int_0^{1-e^{1/\tau}} f_y(y)\,dy = 1$. The same computations for input with Gaussian pdf with a shift of x_o yields

$$f_y(y) = \frac{\tau}{\sqrt{2\pi}\,\sigma}\frac{1}{1-y} \quad \exp\left[-\frac{1}{2}(\ln\left(\frac{1}{1-y}\right)^{\tau} - x_o)/\sigma^2\right]^2 \tag{11}$$

The variation of $f_y(y)$ given in (10) and (11) are sketched in figure 4.

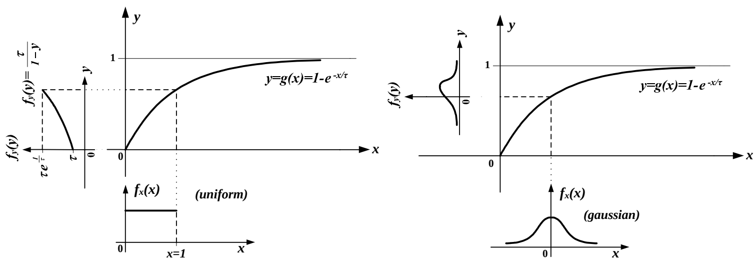

Fig. 4. Uniform probability and Gaussian density functions (pdf) at the model input and the ensuing pdfs at the model output.

The pdf of *u(t)* given by (7) is computed as follows.

$$u(t) = 1 - e^{-[x_2(t) - x_1(t)]/\tau} \qquad (12)$$

We define a new variable w as $w = x_2 - x_1$.

$$f_w(w) = \int_{-\infty}^{+\infty} f_{x_1}(w - x_2) f_{x_2}(-x_2) dx_2$$

We assume x_1 and x_2 have uniform density, as this was the case in our research, $f_w(w)$ is obtained as seen in figure 5.

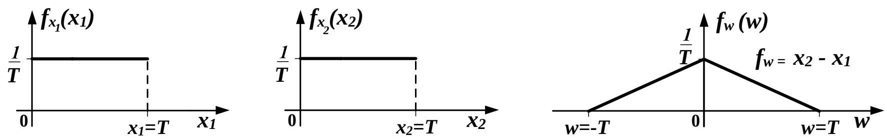

Fig. 5. Probability density function (pdf) of a random variable, which represents the difference of two other random variables with uniform density functions

From figure 5 we note that

$$f_w(w) = -\frac{w}{T^2} + \frac{1}{T} \quad (w > 0) \quad \text{and} \quad f_w(w) = \frac{w}{T^2} + \frac{1}{T} \quad (w < 0) \qquad (13)$$

Using the theorem (8) we obtain for $u \le 0$ and for $u \ge 0$, respectively

$$f_{-u}(u) = \frac{f_w(w_1)}{|g'(w_1)|} = \frac{\frac{\tau}{T^2} \ln(\frac{1}{1-u})^\tau + \frac{\tau}{T}}{1-u} \qquad (14)$$

$$f_{+u}(u) = \frac{f_w(w_1)}{|g'(w_1)|} = \frac{-\frac{\tau}{T^2} \ln(\frac{1}{1-u})^\tau + \frac{\tau}{T}}{1-u} \qquad (15)$$

are obtained, so that

$$\int_{1-e^{T/\tau}}^{0} f_{-u}(u)\, du + \int_{0}^{1-e^{-T/\tau}} f_{+u}(u)\, du = 1 \qquad (16)$$

The input $u(t)$ to nonlinear system is seen in figures 2 and 3. The same calculations for the time varying autoregressive (AR) model coefficient a in (6) yields,

$$f_{a1}(a) = \frac{\tau}{T a}\left[\frac{1}{T}\ln(\frac{1}{a})^\tau + 1\right] \qquad \text{for } a \le 1, \text{ and} \qquad (17)$$

$$f_{1a}(a) = \frac{\tau}{T a}\left[-\frac{1}{T}\ln(\frac{1}{a})^\tau + 1\right] \qquad \text{for } a \ge 1$$

$$\int_{e^{-T/\tau}}^{1} f_{a1}(a)\, da + \int_{1}^{e^{T/\tau}} f_{a1}(a)\, da = 1 \qquad (18)$$

The pdfs of u and a are shown in figure 6 for $\tau=2$ and $T=10$.

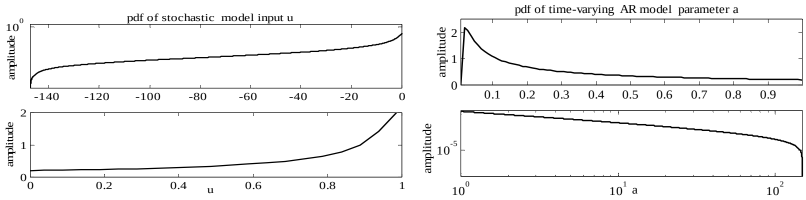

Fig. 6. *Pdf*s of model input *u (left)* and autoregressive model parameter *a(right)*

5 DISCUSSION AND CONCLUSIONS

TS fuzzy modeling is an essential means for the representation of nonlinear dynamic systems for identification, control etc. Such system dynamics are represented by a relatively small number of fuzzy sets compared to other approaches. For nonlinear dynamic system identification, the probability density of stochastic model inputs and outputs can reveal important information about the unknown system. In this respect, the capabilities of fuzzy modeling and its behavior with stochastic excitations are demonstrated in this work. Effective associations of probabilistic data can be made with fuzzy logic and these associations can be exploited in a variety of ways. Exemplary research can be seen in visual perception studies, where a theory of visual perception is developed as given in reference 8.

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